

Light-Front Aspects of Chiral Symmetry

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Abstract

Spontaneous chiral symmetry breaking and axial anomaly are studied in the light-front formulation. The existence of multiple vacua and a Nambu-Goldstone boson, both related to dynamical fermion zero modes, are demonstrated within a simple sigma model with fermions. The Weyl gauge formulation and a consistent gauge invariant point-splitting regularization, which includes the light front time, are crucial for obtaining the anomaly in the massive Schwinger model and QED(3+1).

1 Introduction

Relativistic field theory quantized at the equal light-front time x^+ ($x^\pm = z \pm t$) [1] is both promising and puzzling. Its primary advantage – a kinematical definition of the physical vacuum (modulo zero modes), seems to be in conflict with rich physical contents of the vacuum in the usual space-like quantization. Also, another fundamental property of theories with fermions, namely chiral symmetry [2,3], has not been fully understood in the light-front (LF) framework. For example, non-zero chiral condensates of the Schwinger model have not been obtained within the genuine LF quantum field theory, whose structure differs in many important aspects from the usual space-like theory.

In the present contribution, we will study two aspects of chiral symmetry on the light front: spontaneous symmetry breaking and the axial anomaly. First, we will sketch the relation between the dynamical fermion zero modes and the existence of the Goldstone boson in a simple $O(2)$ -symmetric sigma model with fermions. The axial anomaly will be analyzed in two and four-dimensional quantum electrodynamics in the Weyl-gauge formulation, using a consistent gauge invariant regularization by splitting in LF space *and time* variables.

2 Goldstone theorem on the light front

Spontaneous symmetry breaking is usually associated with non-invariance of the vacuum state of a given Hamiltonian. If there are also non-invariant (composite) operators A with the property $\langle 0|\delta A|0\rangle \neq 0$, existence of a massless state(s) with the quantum numbers of the conserved current(s), corresponding to the symmetry group, can be proved. Due to the simplicity of the physical LF vacuum in the sector of normal modes, this picture can be realized on the LF explicitly in the Fock representation using a finite-volume formulation. The simplest example is a model without gauge symmetry, namely the $O(2)$ -symmetric sigma model with periodic massless fermions [4]. Unitary operators $V(\beta) = \exp(-i\beta Q_5)$, implementing the axial symmetry of the model, contain a part composed of dynamical fermion zero modes b_0, d_0 , and generate a set of degenerate vacua $|\beta\rangle = \exp\left[-i\beta \sum_s 2s[b_0^\dagger(s)d_0^\dagger(-s) + h.c.]\right]|0\rangle$, where s is the LF helicity. There are two non-invariant operators A : $\bar{\psi}\psi = \psi_+^\dagger\gamma^0\psi_- + h.c.$ and $\bar{\psi}\gamma^5\psi = \psi_+^\dagger\gamma^0\gamma^5\psi_- + h.c.$, $A \rightarrow V(\beta)AV^\dagger(\beta) \neq A$, and the conserved axial vector current $j_5^\mu = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi + \text{bosonic part}$ ($\mu = +, -, 1, 2$, see Sec.4 for our notation). Thus, the existence of a massless state $|G\rangle = Q_5|0\rangle = \sum_s 2sb_0^\dagger(s)d_0^\dagger(-s)|0\rangle$ can be derived in a usual way (see [5], e.g.).

3 Axial anomaly in the massive Schwinger model

The LF Hamiltonian of the massive Schwinger model in the gauge $A^- = 0$

$$P^- = \int dx^- \left[\Pi_{A^+}^2 + m \left(\psi_+^\dagger \gamma^0 \psi_- + \psi_-^\dagger \gamma^0 \psi_+ \right) \right] \quad (1)$$

is expressed in terms of the conjugate momentum $\Pi_{A^+} = \partial_+ A^+$ of the A^+ gauge field component and dynamical (+) and dependent (-) fermi fields ψ_\pm . We work in the continuum formulation and all fields are taken to be antiperiodic in x^- [6]. The dynamical component ψ_+ obeys the equation $2i\partial_+\psi_+ = m\gamma^0\psi_-$ and is expanded at $x^+ = 0$ as

$$\begin{aligned} \psi_+(x^-) &= \int_0^\infty \frac{dp^+}{4\pi\sqrt{p^+}} u \left(b(p^+)e^{-\frac{i}{2}p^+x^-} + d^\dagger(p^+)e^{\frac{i}{2}p^+x^-} \right), \quad u^\dagger = (0 \ 1), \\ \{b(p^+), b^\dagger(p'^+)\} &= \{d(p^+), d^\dagger(p'^+)\} = 4\pi p^+ \delta(p^+ - p'^+). \end{aligned} \quad (2)$$

The dependent component ψ_- in (1) satisfies the constraint ($x = (x^+, x^-)$)

$$2i\partial_- \psi_-(x) = m\gamma^0\psi_+(x) + e\psi_- A^+(x), \quad (3)$$

which is inverted by the antiperiodic Green's function $G_a(x^- - y^-; A^+) = \frac{1}{2}\epsilon_a(x^- - y^-) \exp \left[-\frac{ie}{2} \int_{y^-}^{x^-} dz^- A^+(x^+, z^-) \right]$, where $\epsilon_a(x^-)$ is the sign function:

$$\psi_-(x^+, x^-) = \frac{m}{2i} \gamma^0 \int dy^- G_a(x^- - y^-; A^+) \psi_+(x^+, y^-). \quad (4)$$

For a successful calculation of the axial anomaly, it is crucial to work with a consistent definition of the fermion currents. On physical grounds, the vector current has to be odd under C-parity [7]. This can be achieved by taking an antisymmetrized product of fermi fields which is equivalent to normal ordering and thus removes the short-distance singularity in the product even if a gauge invariant splitting of the arguments of the field operators [8] is performed. On the other hand, there is no C-parity restriction in the case of the axial current $j_5^\mu = \psi^\dagger \gamma^0 \gamma^\mu \gamma^5 \psi$. Its regularized and gauge invariant definition is

$$j_5^\pm(x) = \pm 2 \psi_\pm^\dagger(x^+ + \frac{\delta}{2}, x^- + \frac{\epsilon^-}{2}) \psi_\pm(x^+ - \frac{\delta}{2}, x^- - \frac{\epsilon^-}{2}) e^{-\frac{ie}{2} \int_{x-\epsilon/2}^{x+\epsilon/2} dy^- A^+(y^-)}. \quad (5)$$

The regulators $\epsilon = (\epsilon^-, \delta)$ are set to zero at the end of calculations. Now, using the equation of motion for an appropriate choice of split arguments, we get

$$\partial_+ j_5^+(x) = im [\psi_-^\dagger(x) \gamma^0 \psi_+(x) - h.c.] e^{-\frac{ie}{2} \epsilon^- A^+(x)} - \frac{ie}{2} \epsilon^- \partial_+ A^+(x) j_5^+(x). \quad (6)$$

Naively, it looks like the second term above vanishes. Indeed, using the anticommutator (2), one finds $j_5^+ =: j_5^+ : -\frac{e}{2\pi} A^+$ which, however, is not gauge invariant (the second term gets shifted by the residual time-independent gauge transformation). The only way out appears to be the use of a gauge-invariant version of the anticommutator :

$$\{\psi_+(x^-), \psi_+^\dagger(y^-)\} = \frac{1}{2} \Lambda_+ \exp \left(\frac{ie}{2} \int_{x^-}^{y^-} dz^- A^+(z^-) \right) \delta_a(x^- - y^-). \quad (7)$$

and the corresponding change of the anticommutators (2). With this modification, which is analogous to the Schwinger's current definition, the vector current $j^\pm(x) = 2 : \psi_\pm^\dagger(x) \psi(x)_\pm :$ is conserved, $j_5^+(x) =: j_5^+(x) : + \frac{1}{i\pi\epsilon^-}$ and

$$\partial_+ j_5^+(x) = im [\psi_-^\dagger(x) \gamma^0 \psi_+(x) - \psi_+^\dagger(x) \gamma^0 \psi_-(x)] - \frac{e}{2\pi} \partial_+ A^+(x). \quad (8)$$

For the case of the minus ("bad") component of the vector current, we use the constraint (3) for the appropriate choice of arguments and expand in ϵ^-, δ :

$$\partial_- j_5^-(x) = im[\psi_-^\dagger(x)\gamma^0\psi_+(x) - h.c.]e^{-\frac{ie\epsilon^-}{2}A^+(x)} + \frac{ie}{2}\delta\partial_+A^+(x)j_5^-(x). \quad (9)$$

The second term can give a non-zero contribution if the j_5^- current has a singularity δ^{-1} . To investigate this point, we insert ψ_- (4) into the definition of j_5^- . The time dependence of the field $\psi_+^\dagger(x^+ + \frac{\delta}{2}, x^-)$ is assumed to be of the free-field form $\exp(\pm \frac{i}{2}\hat{p}^-\delta)$, where $\hat{p}^- = \frac{m^2}{p^+}$. Normal-ordering j_5^- and using the gauge invariant anticommutator (7), we find for the contraction part

$$\langle j_5^-(x) \rangle = -m^2 \int_0^\infty \frac{dp^+}{p^{+2}} e^{-\frac{i}{2}p^+x^- - \frac{i}{2}\frac{m^2}{p^+}\delta} = -\frac{1}{i\pi\delta}, \quad (10)$$

where the expansion of the modified Bessel function $K_1(m\sqrt{\epsilon^-\delta})$ for a small argument has been used. Thus, the mass dependence completely cancels [9] and the second term in Eq.(9) gives a finite contribution. In this way, the divergence of the axial vector current in the massive LF Schwinger model in the Weyl gauge is

$$\partial_\mu j_5^\mu = \partial_+ j_5^+ + \partial_- j_5^- = 2im[\psi_-^\dagger\gamma^0\psi_+ - \psi_+^\dagger\gamma^0\psi_-] - \frac{e}{\pi}\partial_+A^+. \quad (11)$$

This result includes also the massless case. The anomaly (11) can be written in the usual covariant form $\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$ ($\epsilon^{+-} = -2, A^- = 0$).

4 Anomaly of the LF QED(3+1)

The calculation of the axial anomaly in the four-dimensional $U(1)$ theory proceeds essentially as in the two-dimensional model. In particular, one has to use a four-dimensional analogue of the “gauge-corrected” anticommutator (7). The obvious difference is the presence of the two perpendicular gauge field components $A^i(x)$ (interpreted as a classical background field [10]) and the spin degree of freedom of the fermi field. The field equations are

$$2i\partial_+\psi_+ = (m\gamma^0 - i\alpha^i\partial_i)\psi_- - e\alpha^iA^i\psi_-, \quad (12)$$

$$2i\partial_-\psi_- = (m\gamma^0 - i\alpha^i\partial_i)\psi_+ - e\alpha^iA^i\psi_+ + eA^+\psi_-. \quad (13)$$

The solution of the constraint (13) is again given by the Green's function $G_a(x^- - y^-; A^+)$: $\psi_-(x) = \int \frac{dy^-}{4i} G_a(x^- - y^-; A^+) [m\gamma^0 - i\alpha^i\partial_i - e\alpha^iA^i(y)]\psi_+(y)$. Our notation is $\underline{x} = (x^-, x^\perp)$, $y = (x^+, y^-, x^\perp)$, $x^\perp = x^i$, $\alpha^i = \gamma^0\gamma^i$, $i = 1, 2$. Let us calculate the divergence of the axial vector current

$$j_5^\mu(x) = \psi^\dagger(x^+ + \frac{\delta}{2}, \underline{x} + \frac{\epsilon}{2})\gamma^0\gamma^\mu\gamma^5\psi(x^+ - \frac{\delta}{2}, \underline{x} - \frac{\epsilon}{2}) \exp(-ie \int_{x^+ + \epsilon/2}^{x^+ + \epsilon/2} dy_\alpha A^\alpha(y)):$$

$$\begin{aligned}
\partial_\mu j_5^\mu(x) &= 2im : \left[\psi_-^\dagger(x) \gamma^0 \gamma^5 \psi_+(x) - h.c. \right] : + im C_m - im \overline{C}_m + C + \overline{C} \\
&+ C^i A^i(x + \frac{\epsilon}{2}) - \overline{C}^i A^i(x - \frac{\epsilon}{2}) - ie\epsilon^\mu \left[\partial_i A_\mu \langle j_5^i(x) \rangle + \partial_- A_\mu(x) \langle j_5^- \rangle \right] \\
&+ \frac{ie}{2} \langle j^- \rangle \left[A^+(x + \frac{\epsilon}{2}) - A^+(x - \frac{\epsilon}{2}) \right]
\end{aligned} \tag{14}$$

Here, the field equations and normal ordering have been used to isolate the contractions $C_m = \langle \psi_+^\dagger(x + \frac{\epsilon}{2}) \gamma^0 \gamma^5 \psi_-(x - \frac{\epsilon}{2}) \rangle$, $C = \langle \psi_+^\dagger(x + \frac{\epsilon}{2}) \alpha^i \gamma^5 \partial_i \psi_-(x - \frac{\epsilon}{2}) \rangle$, $C^i = -ie \langle \psi_+^\dagger(x + \frac{\epsilon}{2}) \alpha^i \gamma^5 \psi_-(x - \frac{\epsilon}{2}) \rangle$, with $\overline{C} = C^*(-\epsilon)$, etc. They can be evaluated by simple spinor identities. C_m vanishes and the remaining contractions lead, due to the splitting in x^+ , to Bessel-type integrals, as for example

$$\int \frac{d^3 \underline{p}}{k^+ - p^+} e^{-\frac{i}{2} p^+ \epsilon^- + i p^\perp x^\perp - \frac{i}{2} \frac{(m^2 + p_\perp^2)}{p^+} \delta} = -\frac{i}{8\pi} \frac{k^+}{\delta} e^{-\frac{i}{2} k^+ (\epsilon^- + \frac{\epsilon^2}{\delta}) + \frac{i}{2} \frac{m^2}{k^+} \delta}. \tag{15}$$

Expanding the gauge potentials in δ , one finds finite contributions to the divergence of the axial vector current. From the $C^i A^i$ terms, e.g., we get $e^2/(2\pi^2) \epsilon^{ij} \partial_- A^j \partial_+ A^i$ ($\epsilon^{ij} = -\epsilon^{ji}$) which is a part of the complete result. Indeed, the full abelian anomaly in the covariant form is [10]

$$\partial_\mu j_5^\mu(x) = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{16}$$

Its LF form in the $A^- = 0$ gauge is $e^2/(2\pi^2) \epsilon^{ij} [\partial_+ A^+ \partial_i A^j + \partial_+ A^i \partial_- A^j + \frac{1}{2} \partial_+ A^i \partial_j A^+]$. The second term coincides with our partial result above. The complete treatment with all technical details will be given elsewhere.

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